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Two-dimensional Fokker–Planck solutions and Grassmann variables

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Abstract. After a short outline of the factorization and Grassmann picture of the one-dimensional (1D) Fokker–Planck (FP) equation, we consider a class of spatially inhomogeneous solutions of the 2D FP equation with symmetric 2D (super)potentials. We show that the spatial inhomogeneities of that class of solutions can be attributed to underlying Grassmannian pseudo-degrees of freedom. Such an interpretation may also be applied to FP solutions in three and more dimensions.

1. Introduction

Supersymmetric techniques are now widely used in a rich spectrum of physical problems, covering such diverse fields as quantum gravity, quantum cosmology, particle physics, quantum field theory and statistical mechanics. In the latter area, Grassmann variables were first considered by Parisi and Sourlas [1] when they cast into supersymmetric form a simple model of a scalar field coupled to a random external source. These intriguing variables have been most often used for Langevin equations with complex actions (Langevin equation formulation of quantum field theory [2]), but lately their application has been considerably extended. As remarked by Sourlas [3] the anticommuting variables are introduced more for combinatorial and other technical reasons and do not correspond to spin degrees of freedom as one might naively think. On the other hand, in the context of Brownian diffusion in a 1D bistable potential [4] the simple Witten supersymmetric procedure [5] has led to a remarkably elegant way of computing the smallest non-vanishing eigenvalue λ_1, which is known to characterize the relaxation rate towards equilibrium of a stochastic system.

In this work, our aim is to show that a class of solutions of the 2D FP equation with symmetric 2D superpotential may be interpreted in terms of an underlying Grassmannian structure. The formalism is essentially that employed in quantum supercosmology [6] where, however, as a rule, the cosmological potential is given and one works out the corresponding superpotential. The FP situation is exactly the opposite. The formalism can be easily generalized to more coordinates.

The organization of the paper is as follows. In the next section we outline Witten's scheme [5] for the 1D FP equation [4] including the superspace extension. In section 3 we present a class of 2D FP solutions that can be traced to Grassmannian pseudo-degrees of freedom and we end with some conclusions.

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2. Factorization of the 1D FP operator

Over a decade ago, Bernstein and Brown [4] provided a simple discussion of the correspondence between the 1D FP equation with an arbitrary potential and Witten's supersymmetric quantum mechanics. The 1D FP equation with constant diffusion coefficient (here normalized to unity) and potential drift is

$$\frac{1}{\gamma}\frac{\partial}{\partial t}\mathcal{P}(x,t) = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x} + \sqrt{\gamma}U'(x)\right]\mathcal{P}(x,t) \tag{1}$$

where $U' = \mathrm{d}U/\mathrm{d}x$ is the drift force up to a sign, and $\gamma = 1/k_{\mathrm{B}}T$ in the case of the approach to thermal equilibrium [4], whilst $\gamma = 1/\nu m$ (ν being the collision frequency and m the molecular mass), in chemical reaction kinetics for a system with two stable states, e.g. the *trans-gauche* isomerization process [7]. For more algebraic symmetry, we introduced an overall $1/\gamma$ factor in equation (1), which can always be cast into the Schrödinger equation as follows. Any initial time-dependent distribution $P(x,t)$ will relax at asymptotic times to the static solution

$$\mathcal{P}_{\mathrm{st}}(x) = \mathrm{const} \times \exp[-\gamma U(x)] \tag{2}$$

where const is a normalization constant. The evolution at intermediate times can be discussed conveniently by means of the celebrated ansatz

$$\mathcal{P}(x,t) = \varphi(x,t)\exp(-\tfrac{1}{2}\gamma U(x)). \tag{3}$$

$\mathcal{P}(x,t) \to \mathcal{P}_{\mathrm{st}}$ when $t \to \infty$. It turns the FP evolution of $\mathcal{P}$ into a Schrödinger evolution for φ in imaginary time $(1/\gamma)(\partial\varphi/\partial t) = -H_{\mathrm{FP}}\varphi$, where the FP Hamiltonian is a Hermitian and positive semidefinite operator. It is now easy to proceed with the factorization and the whole Witten scheme. We write $H_{\mathrm{FP},1} = A^{\dagger}A$ with

$$A = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x} - \sqrt{\gamma}U' \qquad \text{and} \qquad A^{\dagger} = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x} + \sqrt{\gamma}U'.$$

Thus, the FP superpotential is proportional to the drift force, and the superpartner Hamiltonian will be $H_{\mathrm{FP},2} = AA^{\dagger}$. The two FP Hamiltonian partners are defined as usual as

$$-H_{\mathrm{FP},1,2} = \frac{1}{\gamma}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{1,2} \tag{4}$$

with the potentials $V_{1,2}$ entering simple Riccati equations $V_{2,1} = -\gamma U'^2 \pm U''$. As is well known the great advantage of the supersymmetric procedure for the FP problem is to replace bistable 'bosonic' potentials with much simpler single-well 'fermionic' ones [4]. The equilibrium distribution can always be written in terms of the FP superpotential as

$$\mathcal{P}_{\mathrm{st}}(x) = \mathrm{const} \times \exp\left(-2\int W_{\mathrm{FP}}(x)\,\mathrm{d}x\right). \tag{5}$$

The time dependence of φ can be exponentiated $\varphi(x,t) = \varphi(x)\exp(-\lambda t)$ [8] leading to the stationary Schrödinger equation

$$\frac{1}{\gamma}\frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2} + [\lambda + V_{1,2}(x)]\varphi(x) = 0. \tag{6}$$

In the following we shall pay particular attention to the spatial function $\varphi(x)$. We would like to exploit some features of the formalism that may arise when considering it as a superfield. We shall make use of some simple rules of the Grassmannian calculus as given in Berezin [9].

To introduce the superspace extension of the 1D FP equation, one should write the FP supercharges as follows

$$Q = \psi\left[-\mathrm{i}\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x} + \mathrm{i}\sqrt{\gamma}\frac{\partial U}{\partial x}\right] \tag{7a}$$

and

$$\bar{Q} = \bar{\psi}\left[-\mathrm{i}\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x} - \mathrm{i}\sqrt{\gamma}\frac{\partial U}{\partial x}\right] \tag{7b}$$

where $\psi = \partial/\partial\theta^0$ and $\bar{\psi} = \theta^0$ in the Grassmann representation. The superspace 1D FP Hamiltonian operator will be

$$H_{\mathrm{FP}} = \{Q, \bar{Q}\} = \left[P^2 + \left(\frac{\partial U}{\partial x}\right)^2 - \frac{\partial^2 U}{\partial x^2}\right] + \theta^0\frac{\partial}{\partial\theta^0}\frac{\partial^2 U}{\partial x^2} \tag{8}$$

where $P = \mathrm{i}(1/\sqrt{\gamma})(\partial/\partial x)$ is the FP momentum operator.

Since we are interested in possible spatial inhomogeneities of 'fermionic'-Grassmann origin we shall expand the φ-superfield in Grassmann variables as follows

$$\varphi(x, \theta^0) = A_+(x) + B_0(x)\theta^0. \tag{9}$$

The conditions $Q\varphi = 0$ and $\bar{Q}\varphi = 0$ defining the 'ground state' give $A_+ = a_+\exp(-\gamma U)$ and $B_0 = b_0\exp(\gamma U)$. In other words, A_+ is required to be 'square-integrable' in superspace, and therefore $U(x) \to \infty$ as $x \to \pm\infty$. Thus, one can see that the B_0 component arising from the Grassmann variable can be discarded for physical reasons. However, had we considered two or more Grassmann variables the conclusion would have been different, as will be seen in the next section.

3. 2D FP solutions and Grassmann variables

We now show that there exist solutions of the 2D FP equation with spatial components that can be naturally attributed to Grassmannian variables. Consider the following probability

$$\mathcal{P} = [a_+^2 + g_0^2(x) + g_1^2(y)]\exp(-2\gamma U) + [a_-^2]\exp(+2\gamma U) \tag{10}$$

where g_0 and g_1 are arbitrary functions of x and y, respectively. Suppose $\mathcal{P}$ is a solution of a 2D FP equation. Then the term with the positive exponent is discarded as not physical. The solution given by equation (10) can also be interpreted as a solution of the 2D FP equation in superspace.

In the 2D case, the supercharges read

$$Q = \psi^\mu\left[-P_\mu + \mathrm{i}\sqrt{\gamma}\frac{\partial U}{\partial q_\mu}\right] \tag{11a}$$

and

$$\bar{Q} = \bar{\psi}^\nu\left[-P_\nu - \mathrm{i}\sqrt{\gamma}\frac{\partial U}{\partial q_\nu}\right]. \tag{11b}$$

The FP momentum operators are

$$P_\mu = \mathrm{i}\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial q_\mu}.$$

With $\{\psi^\mu, \bar\psi^\nu\} = \eta^{\mu\nu}$, where the metric is $\eta^{\mu\nu} = \mathrm{diag}(1,1)$, i.e. $\psi^\mu = \eta^{\mu\nu}\partial/\partial\theta_\nu$ and $\bar\psi^\nu = \theta^\nu$, one will find the superspace FP Hamiltonian to be written in the form

$$H_{\mathrm{FP}} = \{Q, \bar Q\} = \eta^{\mu\nu}\left[P_\mu P_\nu + \frac{\partial U}{\partial q_\mu}\frac{\partial U}{\partial q_\nu} - \frac{\partial^2 U}{\partial q_\mu \partial q_\nu}\right] + \bar\psi^\nu\psi^\mu \frac{\partial^2 U}{\partial q_\mu \partial q_\nu}. \quad (12)$$

The 2D Grassmann representation of the φ field reads

$$\varphi = A_+ + B_0\theta^0 + B_1\theta^1 + A_-\theta^0\theta^1. \quad (13)$$

The ground-state amplitudes are determined by the conditions $Q\varphi = 0$ and $\bar Q\varphi = 0$ with the Q's and φ substituted from equations (11a, b) and equation (13), respectively. Using the ansatz [6]

$$B_\mu = \frac{\partial f}{\partial q_\mu}\exp(-\gamma U) \qquad \mu = 0, 1$$

these conditions lead to the following equation for f:

$$\Box_2 f - 2\gamma \nabla_2 U \nabla_2 f = 0 \quad (14)$$

where

$$\Box_2 = \eta^{\mu\nu} P_\mu P_\nu \quad (15)$$

and

$$\nabla_2 U \nabla_2 f = \eta^{\mu\nu}\frac{\partial U}{\partial q_\mu}\frac{\partial f}{\partial q_\nu}. \quad (16)$$

In the case of separable potential functions $U(x, y) = U_0(x) + U_1(y)$ and separable f functions $f = f_0(x) + f_1(y)$ one will obtain equation (14) in the form

$$\frac{\partial^2 f_0}{\partial x^2} - 2\gamma \frac{\partial U_0}{\partial x}\frac{\partial f_0}{\partial x} = -\left(\frac{\partial^2 f_1}{\partial y^2} - 2\gamma\frac{\partial U_1}{\partial y}\frac{\partial f_1}{\partial y}\right) \quad (17)$$

which is in almost separable form. Denoting the f-derivatives by F_0 and F_1, and the U-derivatives by V_0 and V_1 one will get

$$\frac{\partial F_0}{\partial x} - 2\gamma V_0 F_0 = a_1 \quad (18a)$$

and

$$\frac{\partial F_1}{\partial y} - 2\gamma V_1 F_1 = -a_1 \quad (18b)$$

where a_1 is the separation constant. For $a_1 = 0$, one gets easily $F_0 = \mathrm{const} \times \exp(2\gamma U_0)$ and $F_1 = \mathrm{const} \times \exp(2\gamma U_1)$. The solutions for B_0 and B_1 turn out as follows: $B_0 = b_0 \exp[\gamma(U_0 - U_1)]$ and $B_1 = b_1 \exp[-\gamma(U_0 - U_1)]$, where b_0 and b_1 are constants. Thus, the probability will be the square of the probability amplitude in superspace [9]

$$\mathcal{P} = [a_+^2 + b_0^2 \exp(4\gamma U_0) + b_1^2 \exp(4\gamma U_1)]\exp(-2\gamma U) + a_-^2 \exp(2\gamma U) \quad (19)$$

which corresponds to equation (10) above.

When the separation constant is different from zero, one should write down the solutions of equations (18a, b), which are as follows

$$F_0 = a_1 \exp(2\gamma U_0)\int \exp(-2\gamma U_0)\,\mathrm{d}x + b_0 \exp(2\gamma U_0) \quad (20a)$$

and

$$F_1 = -a_1 \exp(2\gamma U_1)\int \exp(-2\gamma U_1)\,\mathrm{d}y + b_1 \exp(2\gamma U_1) \quad (20b)$$

from which the B coefficients are immediately obtained by multiplying with $\exp(-\gamma U)$. When $a_1 = 0$ we recover the previous results.

4. Conclusions

We have shown, on the basis of a simple separable example, that 2D FP spatially inhomogeneous solutions can be related to Grassmannian pseudo-degrees of freedom. Therefore one might think of them not only as a technical detail but also as having direct physical effects. One can easily build higher-dimensional superfields corresponding to even more complicated spatial inhomogeneities of either non-equilibrium thermodynamics or chemical kinetics. In 3D, equation (13) is written as follows

$$\varphi = A_+ + B_\nu \theta^\nu + \tfrac{1}{2}\epsilon_{\mu\nu\lambda} C^\lambda \theta^\mu \theta^\nu + A_- \theta^0 \theta^1 \theta^2 \tag{21}$$

where $\mu, \nu = 0, 1, 2$, and

$$B_\mu = \frac{\partial f}{\partial q_\mu} \exp(-\gamma U) \qquad C_\lambda = \frac{\partial f}{\partial q_\lambda} \exp(\gamma U)$$

and the Euclidian metric is $\eta^{\mu\nu} = (1, 1, 1)$. Equation (14) is transformed into $\Box_3 f \pm 2\gamma \nabla_3 U \nabla_3 f = 0$, where the operators have the same meaning as in equations (15) and (16), except that the 3D metric is used. Equations (11) and (12) are preserved in form but the attached η-metric is 3D.

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